

A perfect fit

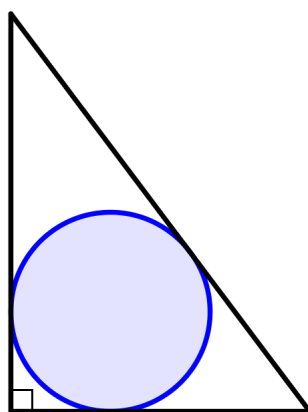
Problem



How can you accurately construct the inscribed circle of a right-angled triangle?



What might be the same and what might be different about your approach to this problem if you were working with graphing software compared to working on paper?



Can you find a right-angled triangle for which the inscribed circle has a radius of 6?

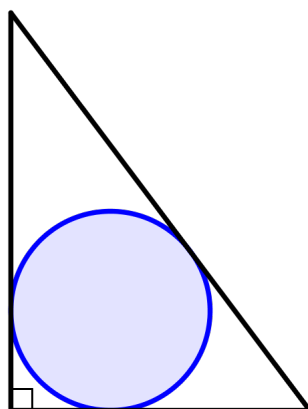
Is your solution unique?

A perfect fit

Suggestion

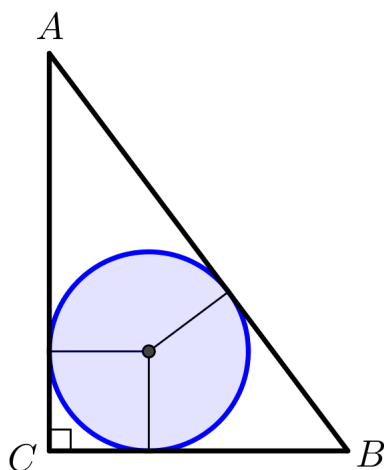


How can you accurately construct the inscribed circle of a right-angled triangle?



Labelling the diagram can often help you to break down a problem and to see what's going on.

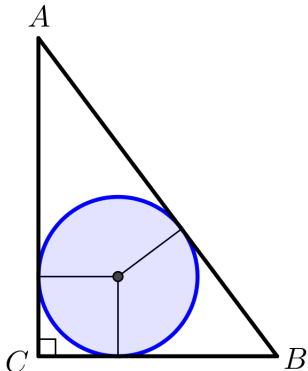
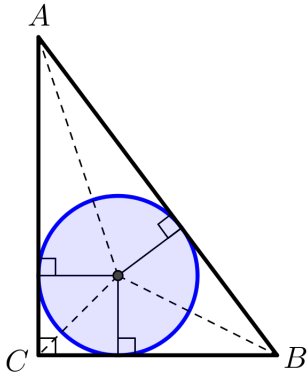
- How could you use the version of the diagram shown below to help?
- What else can you label on this diagram?

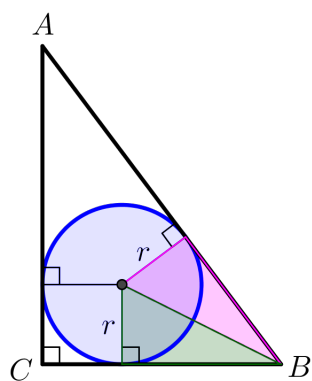
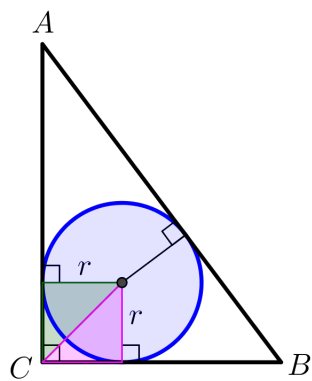
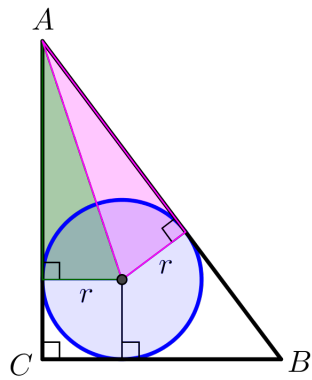


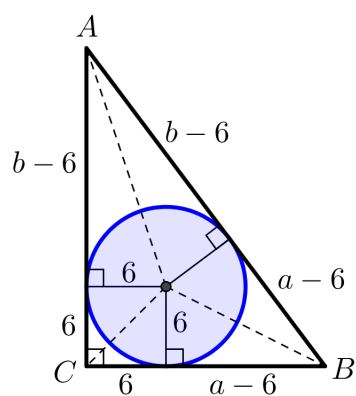
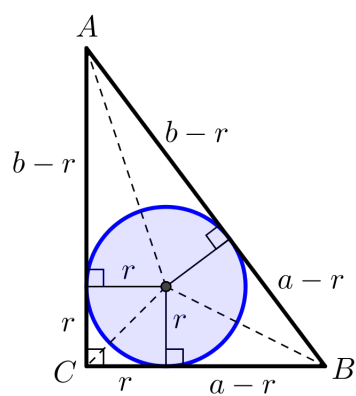
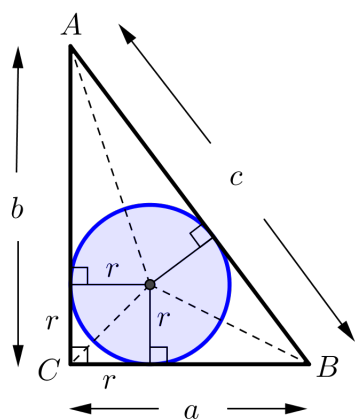
Further suggestion

Can you use this sequence of diagrams to form a chain of mathematical reasoning that will help you to solve the problem?

Try to describe each diagram in turn, using mathematical language where appropriate. A printable version is available [here](#).

Diagram	Description
	
	



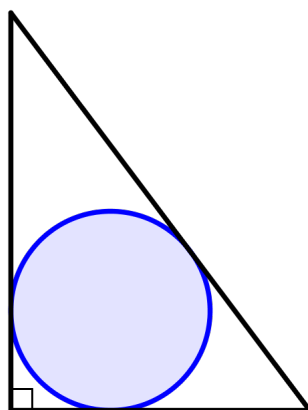


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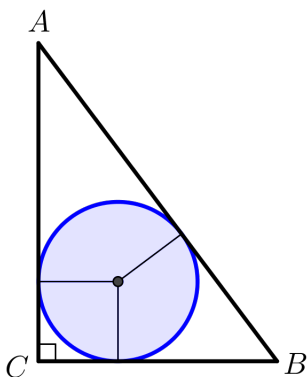
Solution

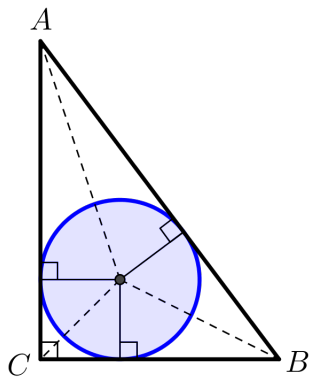


How can you accurately construct the inscribed circle of a right-angled triangle?

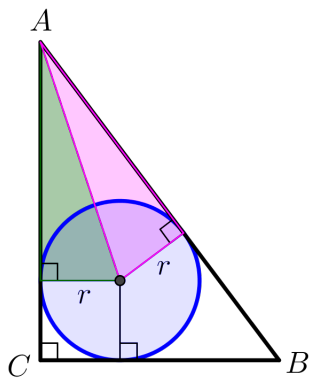


We might start by thinking about the sequence of labelled diagrams shown below (these are suggested in the [suggestion section](#) and a printable version is available [here](#)).

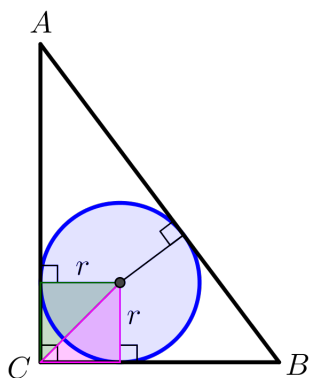
Diagram	Description
	• Mark the centre of the inscribed circle. • Draw lines to show the radii at the points where the circle touches the outer triangle.



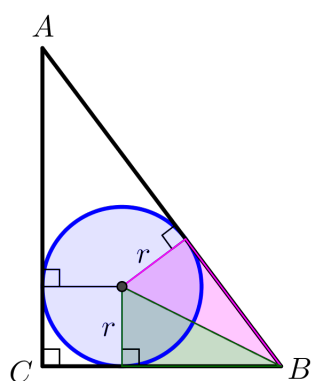
- The outer triangle is tangent to the inscribed circle in three places.
- A tangent meets a radius at a right-angle.
- Draw lines from the centre of the circle to the vertices of the outer triangle, dividing the outer triangle into six, smaller, right-angled triangles.



- The pink triangle is congruent to the green triangle by RHS (they both contain a right-angle, share a common hypotenuse, and a side length of r).
- The pink triangle is also congruent to the green triangle by SSS as they share a side length of r , a common hypotenuse, and a third side length (tangents to a circle from the same point are equal in length).
- The pink and green triangles together form a kite.
- The shared hypotenuse bisects angle A of the outer triangle.



- The pink triangle is congruent to the green triangle by RHS.
- The pink and green triangle are both isosceles (they each have two sides of length r) and together they form a square.
- The shared hypotenuse bisects angle C of the outer triangle.



- The pink triangle is congruent to the green triangle by RHS.
- The pink and green triangles together form a kite.
- The shared hypotenuse bisects angle B of the outer triangle.



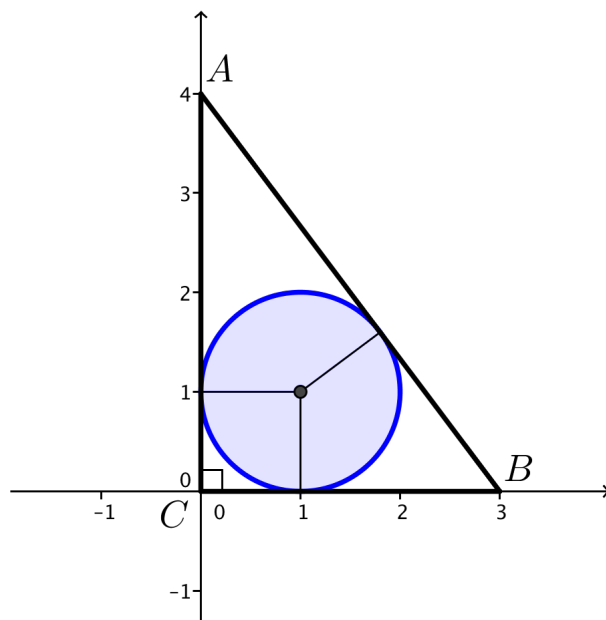
At this point we can summarise that the centre of the inscribed circle lies at the intersection of the angle bisectors of the outer triangle.

In order to accurately construct the inscribed circle we must therefore first construct any two of the angle bisectors of A , B , and C . If we then construct a perpendicular line from the centre to one edge of the outer triangle, we can use this to set the radius on our pair of compasses and hence accurately construct the inscribed circle.



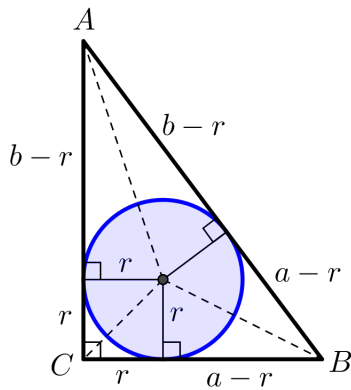
Many examples of graph drawing software will allow you to directly construct angle bisectors and perpendicular lines, so it would be possible to approach the problem in the same way as on paper.

If this is not possible then we would need to think about the coordinate geometry of the problem, identifying the equation of the inscribed circle. We might choose to do this for a specific example before trying to find a general solution.



Can you find a right-angled triangle for which the inscribed circle has a radius of 6?

Diagram	Description
	• Using the fact that we have a square in the bottom left-hand corner of the outer triangle, we can label the length r on the outer triangle. • We will label the sides of the outer triangle a, b and c in the usual way.

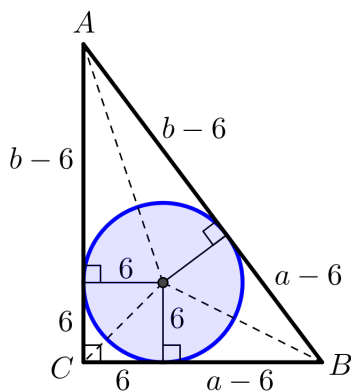


- We can now label the other 'portions' of the perpendicular sides of the triangle $a - r$ and $b - r$ respectively.
- Because we know that we have pairs of congruent triangles, we can also label the two 'portions' of side length c , the hypotenuse of the outer triangle.
- We can write down an expression for side-length c :

$$c = (a - r) + (b - r),$$

which can be simplified to

$$c = a + b - 2r.$$



- We have been asked about the specific case where $r = 6$ so we can substitute this into our diagram and our expression for side-length c :

$$c = a + b - 12.$$

Now, we also require that outer triangle ABC is right-angled so we have the second condition that

$$c^2 = a^2 + b^2.$$

If we substitute for c in the above equation we have

$$(a + b - 12)^2 = a^2 + b^2,$$

which can be expanded to

$$a^2 + b^2 + 144 + 2ab - 24a - 24b = a^2 + b^2,$$

and simplified to

$$72 + ab = 12(a + b).$$



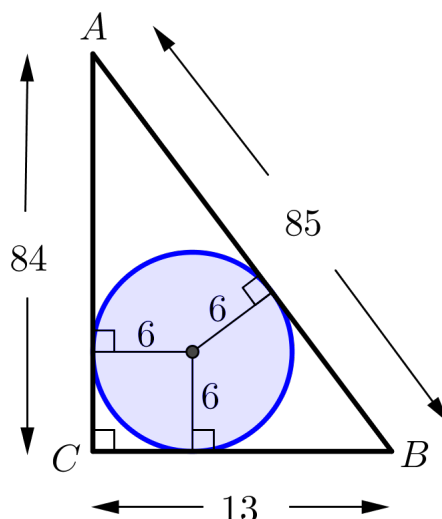
We still have two unknown values in this equation, a and b .

What happens if we assign a value to either a or b and use this to calculate the other?

Rearranging the above equation to make b the subject we have

$$b = \frac{12(a - 6)}{a - 12}.$$

If we choose $a = 13$ then we obtain the following solution to the problem.



Does it matter exactly which value we select for a or b ?

Does the value we select need to be an integer?

Why might we have chosen $a = 13$?

How does the denominator in our formula for b relate to the diagram, and what can it tell us about the values that we select?



Is your solution unique?

You may have some feeling for this already from the work done above.

You might like to experiment with the interactive diagram below in order to clarify your thinking.



- What stays the same and what changes as you move point B ?
- What constraints are there on the base length of the triangle (line segment BC)?
- What constraints are there on the vertical height of the triangle (line segment CA)?
- How will your findings generalise to a right-angled triangle with an inscribed circle of radius r ?



In our example above we chose the base length to be 13. It turns out that the other side lengths are also integer values (84 and 85). This is an example of a Pythagorean Triple.

- How many integer solutions to this problem are there?



Can you find a right-angled triangle for which the inscribed circle has a radius of 6?

Is your solution unique?

In the [solution](#) we substituted the specific case of $r = 6$ into our equations at a fairly early stage.

However, we might have asked ourselves, why $r = 6$? What about other values of r ? Presumably we can take the same approach for any radius but if we have to return to the early stages of working for each specific case that we want to consider then we aren't being very efficient.

Working algebraically

We may therefore have chosen not to substitute for r , but to work more generally.

We would have the two equations

$$c = a + b - 2r \tag{1}$$

and

$$c^2 = a^2 + b^2 \tag{2}$$

from our earlier work and, as before, we can substitute for c in equation (2).

This would give

$$a^2 + b^2 = (a + b - 2r)^2$$

which can be expanded and then simplified to

$$b = \frac{2r(a - r)}{a - 2r}.$$

We could then simply substitute any given value of r into this formula and follow the same procedure of choosing a value for a in order to determine lengths b and c .

Thinking about scaling

Alternatively, we may have chosen to think about the simplest case where $r = 1$. Following through the same approach as before we have that

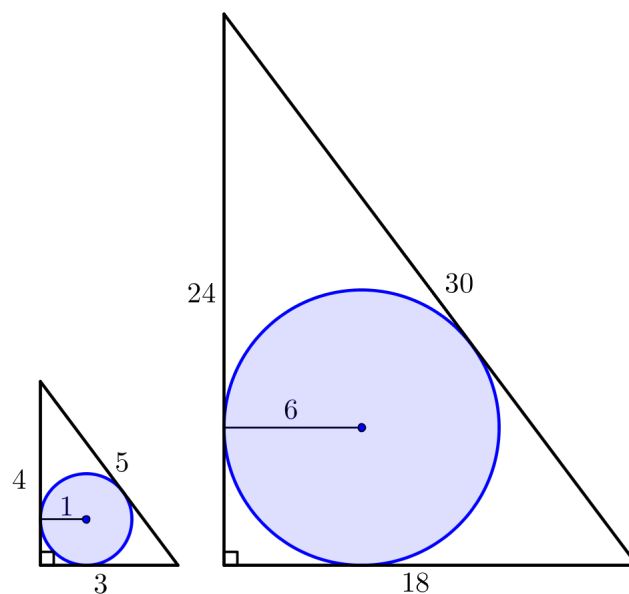
$$b = \frac{2(a-1)}{a-2}.$$

Now, selecting a value for a will generate a solution to this simplest case, for example, if $a = 3$ then we find that $b = 4$ and $c = 5$.



Notice that this is the smallest example of a Pythagorean Triple.

We can then use this simpler 'base case' to find a similar triangle with an inscribed circle of radius 6, we just need to multiply each length by the scale factor of 6, giving us a solution where $a = 18$, $b = 24$, and $c = 30$, as shown below.



The idea of scaling is very powerful. By thinking about this one 'base case' we can very easily generate right-angled triangles for an inscribed circle of any given radius. In the examples illustrated below we have considered integer radii but of course we can also find similar triangles with inscribed circles which have non-integer radii too.

It is worth noticing that changing the selected value of a in our base case will generate another 'family' of solutions. If we refer back to the original problem, this is a way of demonstrating that there is not a unique right-angled triangle for an inscribed circle of a given radius.

